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DELIBERATIONS ON OSCILLATING STRINGS

Johann Bernoulli

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DELIBERATIONS ON OSCILLATING STRINGS

Johann Bernoulli

ABSTRACT 32272

Deliberations on oscillating strings to which small weights are equidistantly attached, where, according to the principles of dynamics, the number of oscillations of the string is sought, expressed in terms of the number of oscillations of a pendulum of given length D. Author

Equal, small oscillating weights C, D, E, F, etc., are equidistantly attached to oscillating string ACDEF, etc. The configuration must be such that each weight reaches position AB in the straight line at the same time. It follows from this condition that the velocities of each weight and the accelerating forces acting on each must be proportional to the distances to be traveled, Cc, Dd, Ee, etc. However, according to the laws of statics, the tension of the string is (to the force by which any weight such as E is impelled toward e) as the sine of the angle DEe to the sine of the angle DEF or IEF, i.e., (because the string is almost straight and the distances of the weights are equal), as the total sine to FI. Therefore, in the same way, the distances Cc, Dd, Ee, etc., are proportional to DG, EH, FI, etc., respectively. The same is true of /13*

*Numbers given in the margin indicate the pagination in the original foreign text.

DG = Gd - Dd = 2Cc - Dd = 2a - x; HE = He - Ee = 2Dd - Cc - Ee = 2x - a - y; FI = If - Ff = 2Ee - Dd - Ff = 2y - x - z; etc.; therefore, 2a - x·a::2x - a - y·x::2y - x - z·y::2z - y - t·z::etc. From this follow the lemmas:

1. When there are two weights, $x = a$, $y = 0$; the others are not considered.

2. When there are three weights, $y = a$, $z = 0$, not considering the others, then 2a - x.a::2x - 2a.x, whence $2ax - x^2 = 2ax - 2a^2$, and $x = a\sqrt{2}$.

3. When there are four weights, $y = x$, $z = a$, and $t = 0$, not considering the others, then 2a - x.a::x - a.x, whence $2ax - x^2 =$ [illegible] $ax - a^2$, and $x =$ [illegible]

4. When there are five weights, $z = x$, $t = a$, $u = 0$, not considering the others, then 2a - x.a::2x - a - y.x::2y - 2x.y; from this arise two equations, $x^2 = a^2 + ay$ and $yx = 2ax$. According to the prior equation, $y = \frac{x^2 - a^2}{a}$, and according to the latter, $y = 2a$; therefore $x = a\sqrt{3}$.

5. When there are six weights, $z = y$, $t = x$, $u = a$, $s = 0$, not considering the others, then 2a - x.a::2x - a - y.x::y - x: y. This yields two equations: $x^2 = a^2 + ay$ and $ay - yx = -ax$. According to the latter equation, $y = \frac{ax}{-a + x}$; according to the other, $y = \frac{x^2 - a^2}{a}$, whence $a^2x = x^3 - ax^2 - a^2x + a^3$, or $x^3 - ax^2 - 2a^2x + a^3 = 0$.

6. When there are seven weights, $t = y$, $u = x$, $s = a$, $w = 0$, not considering the others. Therefore 2 a - x.a::2x - a - y.x::2y - x - z.y::2z - 2y.z; thus we obtain three equations: $x^2 = aa + ay$, $xy = ax + az$ and $xz = 2ay$. From the second equation, $z = \frac{xy - ax}{a}$; according to the third, $z = \frac{2ay}{x}$, from which [illegible] and [illegible]. /15

From the first equation, $y = \frac{x^2 - a^2}{a}$, then $\frac{x^2 - a^2}{a} = \frac{ax^2}{x^2 - 2a^2}$ and consequently, $a^2x^2 = x^4 - 3a^2x^2 + 2a^4$, or $x^4 - 4a^2x^2 + 2a^4 = 0$; so that

$x^2 = 2a^2 \pm a^2\sqrt{2}$, and $x = a\sqrt{2 \pm \sqrt{2}}$; $y = a \pm a\sqrt{2}$;

$z =$ [illegible] $= \pm a\sqrt{4 \pm [illegible]\sqrt{2}}$.

It should be noted here that the lower signs are not squared.

Problem 1

Suppose a string or thread ALB free of all thickness is weighted in the center by weight L, and suppose the thread is kept taut by weight P. The question is the time of a semioscillation for LC. If LC = a, AL or AC = b, and $AL - AC = \frac{AL^2 - AC^2}{AL + AC} = \frac{LC^2}{2AC} = \frac{a^2}{2b}$, and $ALB - AB = \frac{a^2}{b}$ = the descent of weight P tensing the thread; suppose z = the vertical altitude of free fall by which it obtains a velocity equal to that of point L when L arrives at C, which velocity will then be $= \sqrt{z}$. The vis viva (kinetic energy) of the weight L = Lz = the vis viva of the tensing weight $= \frac{a^2}{b} \times P$; therefore, $z = \frac{a^2 \times P}{b \times L}$. However, because the force pulling point L toward C is always proportional to distance LC, it will have to be, assuming the diameter of a circle at its circumference, as I to p; and v, as the velocity of point L to C, the time for LC, or the time for one semioscillation $= \frac{ap}{2v} = \frac{ap}{2\sqrt{z}} = \frac{p\sqrt{b.L}}{2\sqrt{P}}$, and the time for one semioscillation of the pendulum of a given length D, $= p\sqrt{\tfrac{1}{2}D}$. Therefore, $p\sqrt{\tfrac{1}{2}D}$ divided by $\frac{p\sqrt{bL}}{2\sqrt{P}}$, that is, $\frac{\sqrt{2D \times P}}{\sqrt{b \times L}}$ will give the number of oscillations /16 of the thread during one oscillation of the pendulum $D = \frac{\sqrt{4D \times P}}{\sqrt{AB \times L}} = \frac{2\sqrt{D \times P}}{\sqrt{AB \times L}}$.

2. Suppose now thread AFGB is tensed by weight P and weighted by two equal weights, each $= \frac{1}{2}$ L, which divide the thread into three equal parts AF, FG and GB. Suppose also that FC = GE = a and AC = CE = EB = b, and AF - AC - BG - BE $= \frac{a^2}{2b}$, so that AFGB - AB $= \frac{a^2}{b}$ = descent of weight P. Suppose now $\sqrt{z}$ = the velocity of point F to C or point G to E, then the vis viva of the weights F and G alike = Lz = the vis viva of the tensing weight P, $= \frac{a^2}{b} \times P$, from which $z = \frac{a^2 \times P}{b \times L}$. The remainder will be found as previously, except that for the number of oscillations we should have. $\frac{\sqrt{2D \times P}}{\sqrt{b \times L}} = \frac{\sqrt{2D \times P}}{\sqrt{\frac{1}{3} AB \times L}} = \frac{\sqrt{6} D \times P}{\sqrt{AB \times L}}$.

3. If there are three single weights $= \frac{1}{3}$ L, again AF - AC = BH - BI $= \frac{a^2}{2b}$. But FG - CE = HG - IE = (from lemma 2) $\frac{3a^2 - 2a^2\sqrt{2}}{2b}$. From this AFGHB - AB $= \frac{4a^2 - 2a^2\sqrt{2}}{b}$ = the descent of weight P which tenses the thread. Now calling the velocity of point F to C, $\sqrt{z}$, the velocity of point G to E will be $\sqrt{2z}$, from which the quantity of vis viva of all weights at the same time $= \frac{4}{3}z \times L$ = the vis viva of the tensing weight $= \frac{4a^2 - 2a^2\sqrt{2}}{b} \times P$, so that $z = \frac{6a^2 - 3a^2\sqrt{2}}{2b \times L} \times P = \frac{(12a^2 - 6a^2\sqrt{2}) \times P}{AB \times L}$ and so that $\sqrt{z}$ or $v = \frac{\sqrt{12a^2 - 6a^2\sqrt{2} \times P}}{\sqrt{AB \times L}}$.
and the time for FC, or $\frac{ap}{2v} = \frac{p\sqrt{AB \times L}}{2\sqrt{12a^2 - 6a^2\sqrt{2} \times P}}$. Therefore $P\sqrt{\frac{1}{2}D}$ divided by /17
$\frac{ap}{2v}$, that is, $\frac{2\sqrt{6 - 3\sqrt{2}} \times P \times D}{\sqrt{AB \times L}}$, will give the number of oscillations of the thread in one oscillation of the given pendulum D.

4. Suppose there are four weights, each equal to $\frac{1}{4}$L, and assume meanwhile that GE = HI = x, the others remaining as they were. Then again AF - AC = BK - BM $= \frac{a^2}{2b}$; FG - CE = KH - MI $= \frac{x^2 - 2ax + a^2}{2b}$; from this AFGHKB - AB =

$\frac{x^2 - 2ax + 2a^2}{b}$ = the descent of weight P. The velocity of point F to C = $\sqrt{z}$, the velocity of point G to E = $\frac{x}{a}\sqrt{z}$; and in the same way the total of the vis viva of all weights = $\frac{a^2 + x^2}{2a^2} \times z \times L$ = the vis viva of weight P = $\frac{x^2 - 2ax + 2a^2}{b} \times P$; therefore, z = $\frac{2a^2x^2 - 4a^3x + 4a^4 \times P}{a^2 + x^2 \times bL}$. For this reason, $\sqrt{z}$ or v $= \frac{a\sqrt{2x^2 - 4ax + 4a^2 \times P}}{\sqrt{a^2 + x^2 \times bL}}$; and the time for FC = $\frac{ap}{2v} = \frac{p\sqrt{a^2 + x^2 \times bL}}{2\sqrt{2x^2 - 4ax + 4a^2 \times P}}$.

Also, therefore, $p\sqrt{\tfrac{1}{2}D}$ divided by $\frac{ap}{2v}$, i.e., $\frac{\sqrt{4x^2 - 8ax + 8a^2 \times D \times P}}{\sqrt{a^2 + x^2 \times bL}}$ = (because in lemma 3 x = $\frac{1}{2}a + a\sqrt{\frac{5}{4}}$) $\frac{2\sqrt{5 - \sqrt{5} \times D \times P}}{\sqrt{5 + \sqrt{5} \times bL}} = \frac{2\sqrt{25 - 5\sqrt{5} \times D \times P}}{\sqrt{5 + \sqrt{5} \times AB \times L}}$ will yield the number of oscillations of the thread in one oscillation of pendulum D.

5. Suppose there are five weights, each = $\frac{1}{5}$L. Suppose also that GE = KM = x, the remainder always unchanged, HI or y (from lemma 4) = 2a and x = a $\sqrt{3}$, and AF - AC = BN - BO = $\frac{a^2}{2b}$; FG - CE = NK - OM = $\frac{x^2 - 2ax + 4a^2}{2b} = \frac{4a^2 - 2a^2\sqrt{3}}{2b}$, GH - EI = KH - MI = $\frac{y^2 - 2yx + x^2}{2b} = \frac{7a^2 - 4a^2\sqrt{3}}{2b}$; consequently, AFGHKNB - AB = $\frac{12a^2 - 6a^2\sqrt{3}}{b}$ = the descent of weight P. It is further assumed that $\sqrt{z}$ is the velocity of point F to C, the velocity of point G to E = $\sqrt{3z}$, and the velocity of point H to I = 2 $\sqrt{z}$. Following from this the total of vis viva of all weights = $\frac{12}{5}z \times L$ = the vis viva of the tensing weight P = $\frac{12a^2 - 6a^2\sqrt{3}}{b} \times P$, from which z = $\frac{10a^2 - 5a^2\sqrt{3} \times P}{2b \times L} = \frac{30a^2 - 15a^2\sqrt{3} \times P}{AB \times L}$; and in this way, $\sqrt{z}$ or v = $\frac{a\sqrt{30 - 15\sqrt{3} \times P}}{\sqrt{AB \times L}}$, and from this the time for FC or $\frac{ap}{2v} = \frac{p\sqrt{AB \times L}}{2\sqrt{30 - 15\sqrt{3} \times P}}$. Therefore,

$pV\frac{1}{2}D$ divided by $\frac{ap}{2v}$, i.e., $\frac{\sqrt{60 - 30\sqrt{3} \times D \times P}}{\sqrt{AB \times L}}$, will give the number of oscillations of the thread during one oscillation of pendulum D.

6. There shall be six weights, each = $\frac{1}{6}$L. Assuming that now GE = NO = x, HI = KM = y, then AF - AC = BR - BS = $\frac{a^2}{2b}$, FG - CE = RN - OM = $\frac{x^2 - 2ax + a^2}{2b}$, and GH - EI = NK - OM = $\frac{y^2 - 2yx + x^2}{b}$; therefore, AFGHKNRB - AB = $\frac{2x^2 - 2ax + 2a^2 + y^2 - 2yx}{b}$ = the descent of weight P. However, $\sqrt{z}$ is assumed as the velocity of point F to C, the velocity of point G to E = $\frac{x\sqrt{z}}{a}$, and the velocity of point H to I = $\frac{y\sqrt{z}}{a}$; therefore, the total of the vis viva of all weights = $\frac{a^2 + x^2 + y^2}{3a^2} z \times L$ = the vis viva of tensing weight P = /19 $\frac{2x^2 - 2ax + 2a^2 + y^2 - 2yx}{b}$ x P, and thus, $\sqrt{z} = \frac{\sqrt{6a^2x^2 - 6a^3x + 6a^4 - 3a^2y^2 - 6a^2yx \cdot P}}{\sqrt{a^2 + x^2 + y^2 \times bL}} = v.$

From this the time for FC = $\frac{ap}{2v}$ = $\frac{ap\sqrt{a^2 + x^2 + y^2 \times bL}}{2\sqrt{6a^2x^2 - 6a^3x + 6a^4 - 3a^2y^2 - 6a^2yx \times P}}$. For this reason $pV\frac{1}{2}D$ divided by $\frac{ap}{2v}$, i.e., $\frac{\sqrt{12a^2x^2 - 12a^3x + 12a^4 + 6a^2y^2 - 12a^2yx \times D.P}}{a\sqrt{a^2 + x^2 + y^2 \times bL}}$ = (because of lemma 5, where y = $\frac{ax}{-a + x}$ and y = $\frac{x^2 - a^2}{a}$, and from this $y^2 = x^2 + ax$)

$$\frac{\sqrt{18a^2x^2 + 6a^3x + 12a^4 - 12ax^3 \times D \times P}}{a\sqrt{a^2 + 2x^2 + ax \times bL}} = \frac{\sqrt{126ax^2 + 42a^2x + 84a^3 - 84x^3 \times D \times P}}{\sqrt{a^3 + 2ax^2 + a^2x \times AB \times L}}$$

= (because of lemma 5, $x^3 = ax^2 + 2a^2x - a^3$) $\frac{\sqrt{42ax^2 - 126a^2x + 168a^3 \times D \times P}}{\sqrt{a^3 + 2ax^2 + a^2x \times AB \times L}}$ =

$\frac{\sqrt{42x^2 - 126ax + 168a^2 \times D \times P}}{\sqrt{2x^2 + ax + a^2 \times AB \times L}}$ which will give the number of oscillations of the thread during a single oscillation of pendulum D, after which for x would be substituted its value, which is the root of the equation $x^3 - ax^2 - 2a^2x + a^3 = 0$.

Solutions of These Same Problems Based on Principles of Statics

Lemma 1. Suppose the natural gravitational force g, by which bodies /20 are naturally animated, i.e., urged to descend, x would be the distance of the descent, v the velocity at the end of the descent, t the time of the descent, M the mass of the weight P; $M \times g = P$; $\frac{gdx}{v} = dv$; therefore, $\sqrt{2gx} = v$.

Lemma 2. $\frac{dx}{v} = dt = \frac{dx}{\sqrt{2gx}}$; therefore $t = \frac{\sqrt{2x}}{\sqrt{g}}$.

Lemma 3. As shown elsewhere, the natural time of the descent through the diameter of any circle is the time of a semioscillation of a cycloid having the same altitude as the circle, as $I:\frac{1}{2}p::2:p$. Then there will be the time of a semioscillation of a pendulum of given length $D = \frac{p\sqrt{D}}{2\sqrt{g}}$, and, as per the preceding lemma, the time of descent through the diameter $= \frac{\sqrt{D}}{\sqrt{g}}$.

Lemma 4. Point F tends toward C by forces which are proportional to the distance FC, as demonstrated, and from anywhere from which point F begins to be moved, the distance FC is always traveled in equal times. Suppose the force applied in any distance $= f \times FC$ (by f I understand the parameter of this force, considering that an absolutely assumed force can be increased and diminished). Considering this as given, the distance FC, taken from the point when it is at rest, = a, any part FO = x; then $\frac{f.\overline{a-x}.dx}{v} = dv$; further, $v = \sqrt{f.2ax - x^2}$, hence, $dt = \frac{dx}{\sqrt{f.2ax - x^2}}$ and $t = \frac{1}{\sqrt{f}} \int \frac{dx}{\sqrt{2ax - x^2}}$. For this reason, the total of $FC = \frac{p}{2\sqrt{f}}$.

Problem 1. Take AF in the second and following figures. The force /21
of weight P to the force urging point F toward C is as the sine of angle AFC to the sine of angle VFB = sine of angle AFC to the sine of twice angle FAC = (because FAC is considered infinitely small) AC·2FC::b·2a, so that the force by which point F is urged toward C = $\frac{2a}{b}$ x P = $\frac{2a}{b}$ M x g (I call M the mass of weight P). However, since weight L -- the gravity of which is disregarded here, as only its small mass is considered -- must be urged toward C by a force expressed by $f \times a \times L$, then $\frac{2a}{b} M \times g = f a \times L$, from which $f = \frac{2gM}{bL}$; so that, as per lemma 4 of this paper, the time for FC $\left(\frac{p}{2\sqrt{f}}\right) = \frac{p\sqrt{bL}}{2\sqrt{2gM}}$ = the short time of semi-oscillations of the thread; by dividing then the time of a semioscillation of given pendulum D, which (as per lemma 3) $= \frac{p\sqrt{D}}{2\sqrt{g}}$, by the time of the semioscillation of the thread $\frac{p\sqrt{bL}}{2\sqrt{2gM}}$, which yields $\frac{\sqrt{2D \times M}}{\sqrt{b \times L}} =$ (substituting weights for the mass) $\frac{2\sqrt{D \times P}}{\sqrt{AB \times L}}$ giving the number of oscillations of the thread, which is sought, as in the preceding solution by vis viva.

Problem 2. Now P is to the force of point F toward C as the sine AFC to the sine of VFG or the sine of FAC::b.a; from this, the force of point F toward C = $\frac{a}{b} M \times g = f a \times \frac{1}{2} L$, so that $f = \frac{2gM}{bL}$, and the time for FC $\left(\frac{p}{2\sqrt{f}}\right) = \frac{p\sqrt{bL}}{2\sqrt{2gM}}$ = the time of a semioscillation of the thread. When $\frac{p\sqrt{D}}{2\sqrt{g}}$ is divided by $\frac{p\sqrt{bL}}{2\sqrt{2gM}}$ the result is $\frac{\sqrt{2D \times M}}{\sqrt{bL}} = \frac{\sqrt{6 D \times P}}{\sqrt{AB \times L}}$ for the number of oscillations which is sought, as above.

Problem 3. Here and in the following the force of point F toward C is called ϕ, so that $P.\phi :: \int AFC . \int VFG$; (by $\int$ I understand the sine of the

angle). From the second lemma, as mentioned in the previous method, $\int VFG$ /22
(being infinitely small) = 2a - a $\sqrt{2}$, if b is assumed as the radius; therefore,

$\Phi = \frac{2a - a\sqrt{2}}{b} \times P = \frac{2a - a\sqrt{2}}{b} \times Mg = fa \times \frac{1}{3}L$; from this $f = \frac{\overline{6 - 3\sqrt{2}}.Mg}{[illegible]}$, and the time for

FC $\left(\frac{p\,l}{2\sqrt{f}}\right) = \frac{p\sqrt{bL}}{2\sqrt{\overline{6 - 3\sqrt{2}} \times Mg}}$ = the time of a semioscillation of the thread; thus, by

dividing $\frac{p\sqrt{D}}{2\sqrt{g}}$ by $\frac{p\sqrt{b}[illegible]}{2\sqrt{[illegible] - 3\sqrt{2}} \times Mg}$, we obtain $\frac{\sqrt{\overline{6 - 3\sqrt{2}} \times D \times M}}{\sqrt{bL}} = \frac{\sqrt{2[illegible] - 3\sqrt{2}[illegible] D \times P}}{\sqrt{AB \times L}}$, the sought

number, as previously.

Problem 4. Here again P : Φ :: $\int AFC$: $\int VFG$. From lemma 3, for the

above method, $\int VFG$ (because it is infinitely small) = $\frac{3}{2}a - \sqrt{\frac{5}{4}a^2} = \frac{3a - a\sqrt{5}}{2}$

assuming b as the total sine: from this, $\Phi = \frac{3a - a\sqrt{5}}{2b} \times P = \frac{[illegible]\sqrt{5}}{2b} \times Mg = [illegible] \frac{1}{4}L$,

from which $f = \frac{\overline{6 - 2\sqrt{5}} \times Mg}{bL}$, and the time for FC $\left([illegible]\right) \frac{[illegible]}{2\sqrt{6 - 2\sqrt{5}} \times Mg}$ = the time

of a semioscillation of the thread. From here $\frac{p\sqrt{D}}{2\sqrt{g}}$ divided by $\frac{p\sqrt{bL}}{2\sqrt{6 - 2\sqrt{5}}\,Mg}$

will give us $\frac{\sqrt{\overline{6 - 2\sqrt{5}} \times D \times M}}{\sqrt{bL}} = \frac{\sqrt{\overline{30 - 10\sqrt{5}} \times D \times P}}{\sqrt{AB \times L}}$, which will give the number of oscil-

lations, in conformity with the above, which is $\frac{\sqrt{\overline{30 - 10\sqrt{5}} \times D \times P}}{\sqrt{AB \times L}}$, $\frac{[illegible] - 5\sqrt{5} \times D \times P}{\sqrt{5 + \sqrt{5} \times AB \times L}}$

Problem 5. The figure will have to be imagined. Here we have from the

preceding lemma 4 $\int VFG$ (which is infinitely small) = $2a - a\sqrt{3}$; thus,

$\Phi = \frac{2a - a\sqrt{3}}{b} \times P = \frac{2a - a\sqrt{3}}{[illegible]\,b} \times Mg = fa \times \frac{1}{5}L$, then $f = \frac{\overline{10 - 5\sqrt{3}} \times Mg}{bL}$, and the time for FC

$\left(\frac{p}{2\sqrt{f}}\right) = \frac{[illegible]}{2\sqrt{10 - 5\sqrt{3}} \times Mg}$ = the time of a semioscillation of the thread. Therefore, by

dividing $\frac{p\sqrt{D}}{2\sqrt{g}}$ through this, the result is $\frac{\sqrt{\overline{10 - 5\sqrt{3}} \times D \times M}}{\sqrt{bL}} = \frac{\sqrt{\overline{50 - 30\sqrt{3}} \times D \times P}}{\sqrt{AB \times L}}$ for the

number of oscillations sought, as above.

Problem 6. From lemma 5, as previously stated, $\int VFG$ shall be (as it is infinitely small) = 2a - x, where x is the root of the equation $x^3 - ax^2 -$ /23
$2a^2x + a^3 = 0$, then $\Phi = \frac{2a-x}{b} \times P = \frac{2a-x}{b} \times Mg = fa \times \frac{1}{6}L$; from this, $f = \frac{12a-6x}{ab \times L} \times Mg$ and the time for FC $\left(\frac{p}{2\sqrt{f}}\right) = \frac{p\sqrt{ab.L}}{2\sqrt{12a-6x.Mg}}$ = the time of a semioscillation of the thread. By dividing $\frac{p\sqrt{D}}{2\sqrt{g}}$ through this, we obtain $\frac{\sqrt{12a-6x \times M \times D}}{\sqrt{ab \times L}} = \frac{\sqrt{84a-42x \times D \times P}}{\sqrt{a.AB.L}} = \frac{\sqrt{42x^2 - 126ax + 168a^2 \times B \times P}}{\sqrt{2x^2 + ax + a^2 \times D \times P}}$, as will be shown by crosswise multiplication.

Observation

Generally, this subject may be treated for any number of weights. If the number of weights is assumed to be n, and $\Phi = \frac{2a-x}{b} \times Mg = fa \times \frac{1}{n}L$, from this $f = \frac{2na-nx \times Mg}{ab \times L}$, and the time for FC $\left(\frac{p}{2\sqrt{f}}\right) = \frac{P\sqrt{ab \times L}}{2\sqrt{2na-nx \times Mg}}$ = the time of a semi-oscillation of the thread. Therefore, $\frac{p\sqrt{D}}{2\sqrt{g}}$ divided by this will yield

$\frac{\sqrt{2na-nx \times D \times M}}{\sqrt{ab.L}} = \frac{\sqrt{n+1.\ 2na-nx \times D \times P}}{\sqrt{a \times AB \times L}}$ = the sought number of oscillations of the thread during one oscillation of the given pendulum D. In this expression we substitute for x its value, which must be sought by the method used for the previous lemmas. If, for example, there are seven weights, in which case n = 7, and x (as per the preceding lemma 6) = $a\sqrt{2 - \frac{1}{2}\sqrt{2}}$, and $\frac{\sqrt{n+1.\ 2na-nx \times D \times P}}{\sqrt{a \times AB \times L}} =$ $\frac{2\sqrt{28a-14x \times D \times P}}{\sqrt{a.AB.L}} = \frac{2\sqrt{28-14\sqrt{2-\frac{1}{2}\sqrt{2}} \times D \times P}}{\sqrt{AB \times L}}$ = the number of oscillations sought, which will be as near as possible to $\frac{2\sqrt{2}.D \times P}{\sqrt{AB \times L}}$, which should be smaller.

Problem 7. Assume a musical string AB of uniform thickness, of a mass /24
quantity = L, tensed by a weight P = Mg; the number of oscillations is sought during one oscillation of the given pendulum D.

Solution. The string should be outside the straightlined figure AB and cover the curve-lined figure AEB which should be so that, whenever its point K arrives at the same moment at the corresponding point H in the straightlined figure as the median point E arrives at C, the accelerating force by which point K is urged toward H be proportional to the distance KH. When two tangents are drawn nearest to KG, GF, and from K and S, applied to KH and SI, according to the principle of statics weight P or Mg is to the force by which the particle KS of the string is urged toward H, as the sine of the angle KSO which is assumed to be a right angle, to the sine of the angle GKF, that is as I to $\frac{FG}{KG}$; therefore, KF can be considered as being perpendicular to the axis CF and therefore this force to K = $\frac{FG \times Mg}{KG} = f \times KH \times dL$, and from this the accelerating force or $f \times KH = \frac{FG \times Mg}{KG \times dL}$. However, in order to determine $\frac{FG}{KG}$, it must be noted that the curve AEB is an elongated companion to a trochoid, i.e., its nature, described by the quadrant of the circle EMN and the line KR drawn parallel to the basis AC, so that AC•KR::EMN•EM, demonstration of which will be given below. Now assume that EC = a, ER = x, EM = s, EMN = $\frac{1}{2}$pa (I always understand I to p to be as the diameter to the circumference). If AC•EMN::n•I, then KR = ns, and it will be found that the subtangent RG = $\frac{s\sqrt{2ax-x^2}}{a-x}$, CG =

$a - x + \frac{s\sqrt{2ax-x^2}}{a-x}$, and their differential FG = $[illegible] \, dx + \frac{asdx - xsdx}{a\sqrt{2ax-x^2}} [illegible] + dx$ = /25

$\frac{asdx - xsdx}{a\sqrt{2ax-x^2}}$, so that $\frac{FG}{KG}$ or, what is the same, $\frac{FG}{KR} = \frac{adx - xdx}{ns\sqrt{2ax-x^2}}$, and the element KS which is taken as being equal to KO = nds = $\frac{nadx}{\sqrt{2ax-x^2}}$. AB•HI(KO)::L•dL, from this $dL = \frac{KO}{AB} \times L = \frac{nadx}{AB\sqrt{2ax-x^2}} L$; if these are substituted we receive for the accelerating force $\frac{FG \times Mg}{KO \times DL} = \frac{AB \times [illegible] \times Mg}{[illegible] \, a \times L}$ = (because npa = AB)

$\frac{p^2\,\overline{a-x}.Mg}{AB\times L} = \frac{p^2\,KH}{AB\times L}\times Mg = f\times KH$; so that $f = \frac{p^2\times Mg}{AB\times L}$, and the time for KH $\left(\frac{p}{2\sqrt{f}}\right) = \frac{\sqrt{AB\times L}}{2\sqrt{M\times g}}$ = the time of a semioscillation of the musical string. When $\frac{p\sqrt{D}}{2\sqrt{g}}$ is divided by $\frac{\sqrt{AB\times L}}{2\sqrt{M\times g}}$ we obtain $\frac{p\sqrt{D\times M}}{\sqrt{AB\times L}} = \frac{P\sqrt{D\times P}}{\sqrt{AB\times L}}$; being the number of oscillations of the string during one oscillation of the pendulum of given length D; so found Taylor (see Meth. Increm. p. 93). And so have I found using the principle of vis viva, as follows:

Assume DN = x; NG = y = $n\int\frac{a\,dx}{\sqrt{2ax-x^2}}$; DG = s; DC = a.ds - dy = $\frac{ds^2 - dy^2}{ds + dy}$ = $\frac{dx^2}{2dy}$ = (because y = $n\int\frac{a\,dx}{\sqrt{2ax-x^2}}$)$dx^2 : \frac{2na\,dx}{\sqrt{2ax-x^2}} = \frac{dx\sqrt{2ax-x^2}}{2na}$; and thus DA - CA = $\int\frac{dx^2}{2dy} = \int\frac{dx\sqrt{2ax-x^2}}{2na} = \frac{\frac{1}{2}CD.DEF}{2nCD} = \frac{\frac{1}{4}DEF}{n} = \frac{DEF}{4n} = \frac{AC}{4n^2} = \frac{AB}{8n^2}$; from here 2DA - 2CA = ADB - AB = $\frac{AB}{4n^2}$ = the difference between the arc and the string.

The radius of the osculum in G, assuming the element Gg or ds as constant, is generally $\frac{ds\,dy}{d^2x}$ = (in the most elongated curves where dy = ds) $\frac{dy^2}{d^2x}$. Therefore, in this case the most elongated companions of the trochoid, where dy = ds = $\frac{na\,dx}{\sqrt{2ax-x^2}}$ = constant, will be $nad^2x\sqrt{2ax-x^2} = \frac{-na^2dx^2 + naxdx^2}{\sqrt{2ax-x^2}} = 0$; from this $d^2x = \frac{\overline{a-x}.dx^2}{2ax-x^2}$, so that the radius of the osculum = $\frac{dy^2}{d^2x}$ = $\frac{n^2a^2dx^2}{2ax-x^2} : \frac{\overline{a-x}.dx^2}{2ax-x^2} = \frac{n^2a^2}{a-x}$, i.e., the radii of the osculum are reciprocal to GH.

Assume now that the weight tensing the string is P, the weight of the string itself AB, L; the velocity of point D, while oscillating, at the time it reaches C = $\sqrt{S}$ (S meaning the space needed by point C toward D to reach this velocity in free-fall descent) and the velocity of any point G toward H = , from this $\frac{\overline{a-x}^2}{2a}S\times\frac{Hb}{AB}\times L = \frac{\overline{a-x}^2}{2a}\times\frac{dy.L.S}{AB} = \frac{\overline{a-x}^2}{a} \times S \times \frac{n\,dx.L}{AB\times\sqrt{2ax-x^2}}$ = the vis viva of particle Gg or Hb of the string toward H = $\frac{nS.L}{a.AB}\times\frac{\overline{a-x}^2\,dx}{\sqrt{2ax-x^2}}$; when this is integrated we obtain $\frac{nS.L}{a.AB}\times\overline{a-x}\sqrt{2ax-x^2}+\int dx\sqrt{2ax-x^2}$ = (for the entire string) $\frac{2n.S.L}{a.AB}\times\frac{1}{2}a\times DEF = \frac{nS.DEF}{AB} = \frac{1}{2}$ L·S = the quantity of vis viva of the entire string. This, however, is equal to the vis viva of weight P descending through $\frac{AB}{4n^2} = P \times \frac{AB}{4n^2}$. Therefore $S = \frac{P\times AB}{2n^2\times L} = \frac{2P.DEF^2}{L\times AB}$ and $\sqrt{S} = DEF\sqrt{\frac{2P}{L.AB}}$. /27

From this the time for $DC = \frac{\sqrt{L\times AB}}{\sqrt{2P}}$. Then, the time of a semioscillation of a simple pendulum whose length is assumed to be $C = \frac{DEF}{DC}\times\sqrt{2C}$; since now these two times are equal, the following equation must be made $\frac{\sqrt{L\times AB}}{\sqrt{2P}} = \frac{DEF\times\sqrt{2C}}{DC}$, from this $C = \frac{DC^2\ AB\times L}{4DEF^2\ P}$. Therefore the number of oscillations of the string during the time of one oscillation of the pendulum of given length $D = \frac{2DEF\times\sqrt{D\times P}}{DC\times\sqrt{AB\times L}}$ = (supposing $\frac{2DEF}{DC} = p$) $\frac{p\sqrt{D\times P}}{\sqrt{AB\times L}}$, as Taylor has it, who calls L and N what I call AB and L.

Here follows his proof which has been affirmed above, the oscillating string ADB (see the preceding figure) furnishes the image of an elongated companion curve of a trochoid.

It has been shown above that the sine of the angle of contact in any point G of the string is proportional to the length GH to be traveled. Retaining the same symbols which we have used above, the sine of the angle of contact =

$\frac{d^2x}{ds}$ = (because the figure is extremely elongated and consequently $ds = dy$ =) $\frac{d^2x}{dy}$, with dy, of course, assumed to be constant, but the length to be traveled $GH = a - x$. Therefore, $\frac{d^2x}{dy}$ to $a - x$ is a constant ratio. If this ratio is /28 taken as dy to $n^2 a^2$, then $\frac{n^2a^2d^2x}{dy} = ady - xdy$, or (divided by dy) $\frac{n^2a^2d^2x}{dy^2} =$ $a - x$. When each member is multiplied by dx, we obtain $\frac{n^2a^2dxd^2x}{dy^2} = adx - xdx$; taking the integrals $\frac{n^2a^2dx^2}{2dy^2} = ax - \frac{x^2}{2}$, or $n^2a^2dx^2 = (2ax - x^2)dy^2$, from which $\overset{2dy}{\frac{nadx}{\sqrt{2ax - x^2}}} = dy$ and $n\int\frac{adx}{\sqrt{2ax - x^2}} = y$. Therefore, y (NG)$\cdot \int\frac{adx}{\sqrt{2ax - x^2}}$ (arc DE)::$n \cdot I$, i.e., NG applied in constant ratio to the arc DE, which ratio is very large. The ratio AC to CD is very large (by hypothesis). Therefore, the ratio AC to quadrant DEF also will be very large. But AC$\cdot$DEF::$n \cdot I$ (as demonstrated), for which reason the proposition is maintained as established.